4'SST Section Copy No. 1 of 3

Hvo L

NACA TN No. 1566

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE

No. 1566

DAMPING IN PITCH AND ROLL OF TRIANGULAR WINGS AT SUPERSONIC SPEEDS

By Clinton E. Brown and Mac C. Adams

Langley Memorial Aeronautical Laboratory
Langley Field, Va.

Washington
April 1948

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE NO. 1566

DAMPING IN PITCH AND ROLL OF TRIANGULAR WINGS AT SUPERSONIC SPEEDS

By Clinton E. Brown and Mac C. Adams

SUMMARY

A method is derived for calculating the damping coefficients in pitch and roll for a series of triangular wings and a restricted series of sweptback wings at supersonic speeds. The elementary "supersonic source" solution of the linearized equation of motion is used to find the potential function of a line of doublets, and the flows are obtained by surface distributions of these doublet lines. The damping derivatives for triangular wings are found to be a function of the ratio of the tangent of the apex angle to the tangent of the Mach angle. As this ratio becomes equal to and greater than 1.0 for triangular wings, the damping derivatives, in pitch and in roll, become constant. The damping derivative in roll becomes equal to one-half the value calculated for an infinite rectangular wing, and the damping derivative in pitch for pitching about the apex becomes equal to 3.375 times that of an infinite rectangular wing.

INTRODUCTION

In reference 1, a straightforward method was found for calculating the lift and the drag due to lift of triangular wings. The present paper extends the method to the calculation of rolling and pitching motions of the wings. The damping coefficients in roll and pitch for the limiting case of very slender wings have been calculated (reference 2). The present theory is not limited by the size of the apex angle, and triangular wings with leading edges ahead of and behind the Mach cone originating at the apex of the wing are treated.

In the present theory, based on the linearized equations of motion, the wing is represented by a doublet distribution which can be shown to be equivalent to a vortex distribution. An integral equation is found which can be easily solved by analogy with known relations for two-dimensional incompressible flow. The pressure distributions presented may be used to calculate the damping coefficients of a limited series of wings for which the trailing edges are cut off so that they lie ahead of the Mach cone springing from their foremost point.

SYMBOLS

x, y, z	coordinates of field point (see fig. 1)
x_1, y_1, z_1	coordinates of a source or doublet
$\beta = \sqrt{M^2 - 1}$	
ϕ	disturbance-potential function
ϕ_0	potential of supersonic source
ϕ_S	potential of supersonic source distribution
ϕ_D	potential of supersonic doublet distribution
ϕ_L	potential of a line of doublets
A	source or doublet strength
C	tangent of half-apex angle
C_L	lift coefficient $\left(\frac{\text{Lift force}}{\frac{1}{2}\rho V^2 S}\right)$
C_m	pitching-moment coefficient $\left(\frac{\text{Pitching moment}}{\frac{1}{2}\rho V^2 S\bar{c}}\right)$
C_l	rolling-moment coefficient $\left(\frac{\text{Rolling moment}}{\frac{1}{2}\rho V^2 Sb}\right)$
ϵ	half of apex angle of wing
$f(\sigma)$	doublet-line-distribution function
$\theta = \frac{y}{x}$	
$\sigma = \frac{y_1}{x_1}$	
c	root chord
$\bar{c}$	mean aerodynamic chord $\left(\bar{c} = \frac{2}{S}\int_0^{b/2} (\text{Local chord})^2\, dy = \frac{2}{3}c\right)$
x_0	point about which the wing pitches

M Mach number

ρ density of fluid

V free-stream velocity

$$C_{L_q} = \frac{\partial C_L}{\partial q\bar{c}/2V}$$

$$C_{l_p} = \frac{\partial C_l}{\partial pb/2V}$$

$$C_{m_q} = \frac{\partial C_m}{\partial q\bar{c}/2V}$$

v_x incremental velocity component in x-direction

P lifting-pressure coefficient $\left(\dfrac{\text{Lifting pressure}}{\frac{1}{2}\rho V^2}\right)$

μ Mach angle $\left(\sin^{-1} \frac{1}{M}\right)$

$$\zeta = \frac{x - \beta^2 y\sigma}{\sqrt{1 - \beta^2\sigma^2}\sqrt{x^2 - \beta^2(y^2 + z^2)}}$$

S wing area

q angular velocity of pitch

p angular velocity of roll

b maximum span of wing

K constant

w z-component of velocity

η small quantity

$E'(\beta C)$ complete elliptic integral $\left(\int_0^{\pi/2} \sqrt{1 - (1 - \beta^2 C^2) \sin^2 n}\ dn\right)$

$F'(\beta C)$ complete elliptic integral $\left(\int_0^{\pi/2} \frac{dn}{\sqrt{1 - (1 - \beta^2 C^2) \sin^2 n}}\right)$

Subscripts:

q	pitching condition
p	rolling condition
i	incompressible

ANALYSIS

Solutions must be found that satisfy the linearized differential equation of a nonviscous compressible fluid written

$$\beta^2 \frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \phi}{\partial z^2} = 0 \tag{1}$$

where x, y, z are Cartesian coordinates (see fig. 1), and ϕ is the disturbance-potential function created by the wing. An elementary solution of this equation known as the potential of a supersonic source may be written

$$\phi_0 = \frac{-A}{\sqrt{(x - x_1)^2 - \beta^2 (y - y_1)^2 - \beta^2 (z - z_1)^2}} \tag{2}$$

The quantity A is the strength coefficient of the source. New solutions may be obtained by superposition of such potentials as shown in reference 3. For example, a distribution of sources over a portion of the xy-plane would give the potential

$$\phi_S = \int_{a_3}^{a_4} \int_{a_1}^{a_2} \frac{-A(x_1, y_1)\, dx_1 dy_1}{\sqrt{(x - x_1)^2 - \beta^2 (y - y_1)^2 - \beta^2 z^2}} \tag{3}$$

where the limits chosen must be such that all sources will be located within the forward Mach cone from the field point (x,y,z). Another solution may now be obtained by differentiation with respect to any of the coordinate directions, that is,

$$\phi_D = \frac{\partial \phi_S}{\partial z}$$

$$= \frac{\partial}{\partial z} \int_{a_3}^{a_4} \int_{a_1}^{a_2} \frac{-A(x_1,y_1)\, dx_1\, dy_1}{\sqrt{(x - x_1)^2 - \beta^2(y - y_1)^2 - \beta^2 z^2}} \tag{4}$$

This solution, however, may be considered the vertical or z-component velocity of the source-distribution potential ϕ_S, and as shown in reference 3

$$\underset{\pm z \to 0}{\phi_D} = \pm \pi A(x,y) \tag{5}$$

The step taken in equation (4) also corresponds to the formation of a doublet potential, that is, ϕ_D represents a distribution of doublets over the xy-plane with strengths proportional to $A(x_1,y_1)$. For any known doublet distribution, the velocity component parallel to the surface in any direction s may immediately be obtained from equation (5)

$$\underset{\pm z \to 0}{v_s} = \frac{\partial \phi_D}{\partial s} = \pm \pi \frac{\partial A}{\partial s} \tag{6}$$

The foregoing results are analogous to incompressible-flow relations and it may be stated in general that for every doublet distribution there is a vortex distribution which will produce a similar flow. The vortex distribution and doublet distribution are directly related by equations (5) and (6). These simple concepts, given first by Prandtl (reference 4), may be used directly to obtain the solution of problems in which the pressure distributions are given, such as airfoils of uniform loading. If the equation of the surface is given and the pressure distribution is required, integral equations must be solved. In certain cases, the problem may be simplified if the form of the final potential is known. In reference 2 the disturbance potential for wings of very low aspect ratio was found to be in the form

$$\phi \propto x^2 f\left(\frac{y}{x}, \frac{z}{x}\right) \tag{7}$$

This form of the potential appears quite logical from the standpoint of satisfying the boundary conditions for steady rolling or pitching. In the following analysis, the assumption of a potential in the form of equation (7) is shown to be correct; however, it should be pointed out that the potential of this type must be restricted to the linearized theory and is not of the same general nature as that of a conical field which exists even in the nonlinear problems.

From equation (7) the doublet distribution over the surface will be in the form

$$A = x_1^2 f\left(\frac{y_1}{x_1}\right) \tag{8}$$

and under the assumptions of the linearized theory the lifting-pressure coefficient is now:

$$\begin{aligned} P &= \frac{4v_x}{V} \\ &= \frac{4\pi x_1}{V}\left[2f\left(\frac{y_1}{x_1}\right) - \frac{y_1}{x_1} f'\left(\frac{y_1}{x_1}\right)\right] \end{aligned} \tag{9}$$

The formation of the integral equation follows the method of reference 1. A potential that represents a line of doublets in the xy-plane at an angle $\tan^{-1}\sigma$ to the x-axis is derived in the form of equation (7). Use is made of the boundary conditions to set up an integral equation that introduces the unknown distribution function $f(\sigma)$. The potential of the doublet line may be obtained by following a procedure similar to that used in obtaining equations (3) and (4), and by substituting the expression for A given in equation (8) into equation (4). The expression obtained in the following equation may be seen to represent a line of doublets along which the doublet strength increases as x^3:

$$\begin{aligned} \phi_L &= \frac{\partial}{\partial z}\int_0^{x'} \frac{-x_1^3\, dx_1}{\sqrt{(x - x_1)^2 - \beta^2(y - \sigma x_1)^2 - \beta^2 z^2}} \\ &= -\frac{\beta^2 z(x - \beta^2\sigma y)}{(1 - \beta^2\sigma^2)^{5/2}}\left(3\coth^{-1}\zeta - \frac{\zeta}{\zeta^2 - 1}\right) \\ &\quad + \frac{2\beta^2 z\sqrt{x^2 - \beta^2(y^2 + z^2)}}{(1 - \beta^2\sigma^2)^2} \end{aligned} \tag{10}$$

where

$$\zeta = \frac{(x - \beta^2\sigma y)}{\sqrt{1 - \beta^2\sigma^2}\sqrt{x^2 - \beta^2(y^2 + z^2)}}$$

and x' is the value of x_1 for which the denominator of the integrand vanishes. The potential of the complete wing may now be obtained by an integration with respect to the dimensionless parameter σ

$$\phi = \int_{-C}^{C} f(\sigma)\phi_L \, d\sigma \tag{11}$$

where $\tan^{-1}C = \epsilon$, the half-apex angle, and $f(\sigma)$ is an unknown distribution function. The z-component velocity w can be written for $\beta \frac{z}{x}$ approaching zero

$$w = \frac{\partial\phi}{\partial z} = -x \int_{-\beta C}^{\beta C} \frac{\beta f(\sigma)\left(1 - \beta^2\sigma\theta\right)}{(1 - \beta^2\sigma^2)^{5/2}} \left[3 \coth^{-1}\zeta - \frac{\zeta}{\zeta^2 - 1} - z \frac{\partial\left(\frac{\zeta}{\zeta^2 - 1}\right)}{\partial z}\right] d(\beta\sigma)$$

$$+ 2x \int_{-\beta C}^{\beta C} \frac{\beta f(\sigma)\sqrt{1 - \beta^2\theta^2}}{(1 - \beta^2\sigma^2)^2} \, d(\beta\sigma) \tag{12}$$

where $\theta = \frac{y}{x}$ for convenience. The boundary conditions for rolling may now be written

$$w = -py$$

or

$$\frac{w}{x} = -p\theta \tag{13}$$

For pitching about the y-axis, there is obtained

$$w = -qx$$

or

$$\frac{w}{x} = -q \tag{14}$$

Introduction of equations (13) and (14) into equation (12) provides integral equations which theoretically can be solved for the unknown function $f(\sigma)$. Simpler relations, however, may be obtained if equation (12) is differentiated twice with respect to θ to obtain the quantity $\frac{\partial^2(w/x)}{\partial\theta^2}$. The method for differentiating is indicated in the appendix and gives

$$\frac{\partial^2(w/x)}{\partial\theta^2} = \lim_{\eta\to 0}\left\{6\sqrt{1-\beta^2\theta^2}\int_{-\beta C}^{\beta(\theta-\eta)}\frac{\beta^3 f(\sigma)\,d(\beta\sigma)}{(\beta\sigma-\beta\theta)^4}\right.$$

$$\left. + 6\sqrt{1-\beta^2\theta^2}\int_{\beta(\theta+\eta)}^{\beta C}\frac{\beta^3 f(\sigma)\,d(\beta\sigma)}{(\beta\sigma-\beta\theta)^4} - 4\sqrt{1-\beta^2\theta^2}\left[\frac{f''(\theta)}{\eta}+\frac{f(\theta)}{\eta^3}\right]\right\} \tag{15}$$

The boundary conditions require the foregoing quantity to be zero for both rolling and pitching with the additional requirements on $f(\sigma)$ that, for rolling, at the point $\theta = 0$

$$(w/x)_p = 0 \tag{16}$$

and, for pitching,

$$\frac{\partial(w/x)_q}{\partial\theta} = 0 \tag{17}$$

Equation (15) now yields, for rolling,

$$\lim_{\eta\to 0}\left\{6\int_{-C}^{(\theta-\eta)}\frac{f(\sigma)_p\,d\sigma}{(\sigma-\theta)^4} + 6\int_{(\theta+\eta)}^{C}\frac{f(\sigma)_p\,d\sigma}{(\sigma-\theta)^4} - 4\left[\frac{f''(\theta)_p}{\eta}+\frac{f(\theta)_p}{\eta^3}\right]\right\} = 0 \tag{18}$$

and, for pitching,

$$\lim_{\eta \to 0} \left\{ 6 \int_{-C}^{(\theta-\eta)} \frac{f(\sigma)_q \, d\sigma}{(\sigma-\theta)^4} + 6 \int_{(\theta+\eta)}^{C} \frac{f(\sigma)_q \, d\sigma}{(\sigma-\theta)^4} - 4 \left[\frac{f''(\theta)_q}{\eta} + \frac{f(\theta)_q}{\eta^3} \right] \right\} = 0 \quad (19)$$

Equations (18) and (19) are identical to the equations that would be obtained for similar boundary conditions on a two-dimensional flat plate if an analogous process of distributing the doublets were followed. (See appendix.) The analogue for the rolling motion of a triangular wing would be a two-dimensional flat plate rotating about its midchord point in a stationary stream. The surface potential distribution and therefore the doublet distribution would be

$$f(\sigma)_p = K_p \sigma \sqrt{C^2 - \sigma^2} \tag{20}$$

For the pitching condition the analogue would be a two-dimensional flat plate in a stream flowing normal to the surface. The potential or doublet distribution would be

$$f(\sigma)_q = K_q \sqrt{C^2 - \sigma^2} \tag{21}$$

These potentials, which can be found in references 2 and 5, satisfy equations (18) and (19) by analogy; however, the conditions of equations (16) and (17) must be shown to be satisfied. For the calculations of $(w/x)_p$ and $\frac{\partial (w/x)_q}{\partial \theta}$, and the evaluation of K_p and K_q, only one value of θ need be considered. This value may conveniently be set equal to zero. For rolling motion, equation (20) indicates the doublet distribution to be antisymmetric. Therefore the value of w/x at $\theta = 0$ must be zero, and the condition of equation (16) is satisfied. For the pitching motion, the doublet distribution is symmetrical about $\theta = 0$ and therefore the quantity $\frac{d(w/x)}{d\theta}$ must be zero at $\theta = 0$ and the condition of equation (17) is satisfied.

The constants K_p and K_q may now be evaluated from the relations obtained in the appendix for $\theta = 0$

$$\frac{w}{x} = -q = \lim_{\eta \to 0} \left[-3K_q \int_{-\beta C}^{\beta C} \frac{\sqrt{\beta^2 C^2 - \beta^2\sigma^2}}{\left(1 - \beta^2\sigma^2\right)^{5/2}} \tanh^{-1} \sqrt{1 - \beta^2\sigma^2}\, d(\beta\sigma) \right.$$

$$+ 2K_q \int_{-\beta C}^{\beta C} \frac{\sqrt{\beta^2 C^2 - \beta^2\sigma^2}}{\left(1 - \beta^2\sigma^2\right)^{2}}\, d(\beta\sigma) + K_q \int_{-\beta C}^{\beta(\theta-\eta)} \frac{\sqrt{\beta^2 C^2 - \beta^2\sigma^2}}{\beta^2\sigma^2\left(1 - \beta^2\sigma^2\right)^{2}}\, d(\beta\sigma)$$

$$\left. + K_q \int_{\beta(\theta+\eta)}^{\beta C} \frac{\sqrt{\beta^2 C^2 - \beta^2\sigma^2}}{\beta^2\sigma^2\left(1 - \beta^2\sigma^2\right)^{2}}\, d(\beta\sigma) - \frac{2\beta C K_q}{\beta\eta} \right] \tag{22}$$

$$\frac{d(w/x)}{d\theta} = -p = \lim_{\eta \to 0} \left[3K_p \int_{-\beta C}^{\beta C} \frac{\beta^2\sigma^2\sqrt{\beta^2 C^2 - \beta^2\sigma^2}}{\left(1 - \beta^2\sigma^2\right)^{5/2}} \tanh^{-1} \sqrt{1 - \beta^2\sigma^2}\, d(\beta\sigma) \right.$$

$$+ 2K_p \int_{-\beta C}^{\beta(\theta-\eta)} \frac{\sqrt{\beta^2 C^2 - \beta^2\sigma^2}}{\beta^2\sigma^2}\, d(\beta\sigma) + 2K \int_{\beta(\theta+\eta)}^{\beta C} \frac{\sqrt{\beta^2 C^2 - \beta^2\sigma^2}}{\beta^2\sigma^2}\, d(\beta\sigma)$$

$$- 3K_p \int_{-\beta C}^{\beta C} \frac{\beta^2\sigma^2\sqrt{\beta^2 C^2 - \beta^2\sigma^2}}{\left(1 - \beta^2\sigma^2\right)^{2}}\, d(\beta\sigma) - K_p \int_{-\beta C}^{\beta C} \frac{\sqrt{\beta^2 C^2 - \beta^2\sigma^2}}{\left(1 - \beta^2\sigma^2\right)}\, d(\beta\sigma)$$

$$\left. - \frac{4\beta C}{\beta\eta} K_p \right] \tag{23}$$

Equations (22) and (23) may be integrated by use of tables (reference 6) to give

$$p = \pi K_p \left[\frac{2 - \beta^2C^2}{1 - \beta^2C^2} E'(\beta C) - \frac{\beta^2C^2}{1 - \beta^2C^2} F'(\beta C)\right] \tag{24}$$

$$q = \pi K_q \left[\frac{1 - 2\beta^2C^2}{1 - \beta^2C^2} C'(\beta C) + \frac{\beta^2C^2}{1 - \beta^2C^2} F'(\beta C)\right] \tag{25}$$

$F'(\beta C)$ and $E'(\beta C)$ are complete elliptic integrals of the first and second kind.

The pressure distribution for the rolling wing may now be obtained from equations (9), (20), and (24) and the pressure coefficient is

$$P = \frac{4xpC^2\theta}{V\left[\frac{2 - \beta^2C^2}{1 - \beta^2C^2} E'(\beta C) - \frac{\beta^2C^2}{1 - \beta^2C^2} F'(\beta C)\right]\sqrt{C^2 - \theta^2}} \tag{26}$$

Integration of the pressures over the wing surface gives the forces and moments acting on the wing. The nondimensional derivative C_{l_p} may then be found

$$C_{l_p} = \frac{-\pi C}{4\left[\frac{2 - \beta^2C^2}{1 - \beta^2C^2} E'(\beta C) - \frac{\beta^2C^2}{1 - \beta^2C^2} F'(\beta C)\right]} \tag{27}$$

In the analysis the pitching axis has been taken at the wing apex; however, in application it is desirable to obtain the pressure distribution and the force and moment coefficients for pitching about any point. A superposition of motions is therefore required. The pitching motion about any point x_0 can be made up of a pure pitching motion about the apex of the wing combined with a vertical translational motion of velocity qx_0. The pressure distribution for this translational motion corresponds to that of a wing at a constant angle of attack of

$-\frac{qx_0}{V}$. (See references 1 and 7.) The pressure distribution for the constant angle of attack $-\frac{qx_0}{V}$ is

$$P = \frac{-4C^2qx_0}{VE'(\beta C)\sqrt{C^2 - \theta^2}} \tag{28}$$

Combining equations (9), (21), (25), and (28) gives for the pressure distribution in the pitching case

$$P = \frac{4qx}{V\sqrt{C^2 - \theta^2}}\left[\frac{2C^2 - \theta^2}{\frac{1 - 2\beta^2C^2}{1 - \beta^2C^2}E'(\beta C) + \frac{\beta^2C^2}{1 - \beta^2C^2}F'(\beta C)} - \frac{x_0C^2}{xE'(\beta C)}\right] \tag{29}$$

Integration of the pressures over the wing surface and formation of the nondimensional derivative yields

$$C_{L_q} = \frac{6\pi C}{\frac{1 - 2\beta^2C^2}{1 - \beta^2C^2}E'(\beta C) + \frac{\beta^2C^2}{1 - \beta^2C^2}F'(\beta C)} - \frac{4\pi Cx_0}{E'(\beta C)\bar{c}} \tag{30}$$

and

$$C_{m_q} = \frac{-6\pi C\left(\frac{9}{8} - \frac{x_0}{\bar{c}}\right)}{\frac{1 - 2\beta^2C^2}{1 - \beta^2C^2}E'(\beta C) + \frac{\beta^2C^2}{1 - \beta^2C^2}F'(\beta C)} + \frac{4\pi Cx_0\left(1 - \frac{x_0}{\bar{c}}\right)}{\bar{c}E'(\beta C)} \tag{31}$$

where $\bar{c}$ is the mean aerodynamic chord.

Calculations of these derivatives for triangular wings having their leading edges outside the Mach cone are most easily made by the source distribution method. In this method, the upper and lower sides of the wing may be considered independent of each other. The source distribution function for the rolling wing is

$$g_p\left(x_1,y_1\right) \propto Ky_1 \tag{32}$$

whereas that for the pitching wing is

$$g_q\left(x_1,y_1\right) \propto Kx_1 \tag{33}$$

The calculation of the pressure distribution is not presented, since the subject of the integration of source distributions has been well covered in reference 3.

The pressure distribution for rolling wings outside the Mach cone has been calculated to be

$$P = \frac{4pC^2x}{\pi V(\beta^2C^2 - 1)^{3/2}}\left[(1 + \beta^2C\theta)\cos^{-1}\frac{1 + \beta^2C\theta}{\beta(C + \theta)} - (1 - \beta^2C\theta)\cos^{-1}\frac{1 - \beta^2C\theta}{\beta(C - \theta)}\right] \tag{34}$$

Integrating the pressures over the wing and expressing the derivative in nondimensional form gives

$$C_{l_p} = -\frac{1}{3\beta} \tag{35}$$

For the pressure distribution due to pitching about the point x_0, a combination of flow patterns must again be used. The pressure distribution of a wing at uniform angle of attack $-\frac{qx_0}{V}$ is (reference 3)

$$P = \frac{4qx_0C}{\pi V\sqrt{\beta^2C^2 - 1}}\left[\cos^{-1}\frac{1 - \beta^2C\theta}{\beta(C - \theta)} + \cos^{-1}\frac{1 + \beta^2C\theta}{\beta(C + \theta)}\right] \tag{36}$$

The pressure distribution for pitching then becomes

$$P = \frac{4qx}{\pi V\beta}\left[\frac{2\beta C\sqrt{1 - \beta^2\theta^2}}{\beta^2 C^2 - 1} + \frac{\beta^3 C^3 - 2\beta C - \beta\theta}{(\beta^2 C^2 - 1)^{3/2}}\cos^{-1}\frac{1 + \beta^2 C\theta}{\beta(C + \theta)}\right.$$

$$\left. + \frac{\beta^3 C^3 - 2\beta C + \beta\theta}{(\beta^2 C^2 - 1)^{3/2}}\cos^{-1}\frac{1 - \beta^2 C\theta}{\beta(C - \theta)}\right]$$

$$- \frac{4qx_0\beta C}{\pi V\beta\sqrt{\beta^2 C^2 - 1}}\left[\cos^{-1}\frac{1 + \beta^2 C\theta}{\beta(C + \theta)} + \cos^{-1}\frac{1 - \beta^2 C\theta}{\beta(C - \theta)}\right] \tag{37}$$

The nondimensional derivatives C_{L_q} and C_{m_q} then become

$$C_{L_q} = \frac{8}{\beta} - \frac{8x_0}{\beta\bar{c}} \tag{38}$$

$$C_{m_q} = -\frac{9 - 8\frac{x_0}{\bar{c}}}{\beta} + \frac{8x_0}{\beta\bar{c}}\left(1 - \frac{x_0}{\bar{c}}\right) \tag{39}$$

DISCUSSION AND CONCLUSIONS

Expressions for the lifting-pressure coefficients over triangular wings in roll are given in equations (26) and (34) and in pitch in equations (29) and (37). Equations (26) and (29) are for wings inside the Mach cone and equations (34) and (37) for wings outside the Mach cone. Typical pressure distributions are shown in figure 2 in which the pressure distributions for the two wings in pitch are for pitching about the apex.

Expressions for the quantities C_{l_p}, C_{L_q}, and C_{m_q} are given in equations (27), (30), and (31), respectively, for the case of the wing inside the Mach cone and in equations (35), (38), and (39) for wings lying outside the Mach cone. It will be seen that the parameters βC_{l_p}, βC_{L_q}, and βC_{m_q} may be expressed as functions of βC where

$$\beta C = \frac{\tan \epsilon}{\tan \mu}$$

The stability derivatives may therefore be plotted against this parameter, to give curves which will hold for all triangular wings at any Mach number. These curves are given in figures 3, 4, and 5. For values of βC approaching zero the values of the derivatives closely approach those given in reference 2 which were based on the assumption of very low aspect ratio.

For values of $\beta C \geqq 1$ (that is, for the wing lying outside the Mach cone), the quantities βC_{l_p} and βC_{m_q} become constant, and equal to $-\frac{1}{3}$ and -1, respectively, (the pitching being about the $\frac{2}{3}c$ point). In comparison, the value of βC_{l_p} and βC_{m_q} for infinite-span, rectangular wings are $-\frac{2}{3}$ and $-\frac{8}{3}$, respectively, (the pitching being about the leading edges).

It should be pointed out that the pressure distributions given in this paper may be used directly to calculate the damping in pitch and roll for wings having trailing edges cut off ahead of the Mach cone, the most interesting of this series being the so-called "arrow wings."

It is apparent that a suction force exists at the leading edges of wings in pitch and roll whenever the leading edges are swept behind the Mach cone. A method for obtaining the values of these suction forces was derived in reference 1.

Langley Memorial Aeronautical Laboratory
National Advisory Committee for Aeronautics
Langley Field, Va., December 12, 1947

APPENDIX

METHOD FOR DIFFERENTIATION OF EQUATION (12)

The expression for w (equation (12)) cannot be used directly when z is set equal to zero because of a troublesome singularity in the term $\frac{\zeta}{\zeta^2 - 1}$ and the occurrence of an indeterminate form under the integral sign. To obtain the value of w on the surface, however, it is possible to integrate and then set z equal to zero. The troublesome parts of equation (12) come from the terms involving $\frac{\zeta}{\zeta^2 - 1}$. These terms, written out, may be integrated as follows:

$$\sqrt{1-\beta^2\theta^2}\int_{-\beta C}^{\beta C}\frac{\beta f(\sigma)\left(1-\beta^2\sigma\theta\right)^2}{\left(1-\beta^2\sigma^2\right)^2}\,d(\beta\sigma)\left\{\frac{1}{(\beta\sigma-\beta\theta)^2+\left(1-\beta^2\sigma^2\right)\frac{\beta^2 z^2}{x^2}}-\frac{2\,\frac{\beta^2 z^2}{x^2}\left(1-\beta^2\sigma\theta\right)^2}{\left(1-\beta^2\theta^2\right)\left[(\beta\sigma-\beta\theta)^2+\left(1-\beta^2\sigma^2\right)\frac{\beta^2 z^2}{x^2}\right]^2}\right\}$$

$$=-\left[\frac{\beta f(\sigma)\left(1-\beta^2\sigma\theta\right)^2\sqrt{1-\beta^2\theta^2}\,(\beta\sigma-\beta\theta)}{\left(1-\beta^2\sigma^2\right)^2\left[(\beta\sigma-\beta\theta)^2+\left(1-\beta^2\sigma^2\right)\frac{\beta^2 z^2}{x^2}\right]}\right]_{-\beta C}^{\beta C}+\sqrt{1-\beta^2\theta^2}\int_{-\beta C}^{\beta C}\frac{d(\beta\sigma)(\beta\sigma-\beta\theta)}{(\beta\sigma-\beta\theta)^2+\left(1-\beta^2\sigma^2\right)\frac{\beta^2 z^2}{x^2}}\,\frac{d\left[\frac{\beta f(\sigma)\left(1-\beta^2\sigma\theta\right)^2}{\left(1-\beta^2\sigma^2\right)^2}\right]}{d(\beta\sigma)} \tag{A1}$$

Introducing the limits and then setting $z = 0$ gives

$$-\frac{\beta f(C)(1-\beta^2\theta C)^2\sqrt{1-\beta^2\theta^2}}{(1-\beta^2C^2)^2(\beta C-\beta\theta)} - \frac{\beta f(-C)(1+\beta^2\theta C)^2\sqrt{1-\beta^2\theta^2}}{(1-\beta^2C^2)^2(\beta C+\beta\theta)} + \sqrt{1-\beta^2\theta^2}\int_{-\beta C}^{\beta C}\frac{d(\beta\sigma)}{\beta\sigma-\beta\theta}\,\frac{d\left[\frac{\beta f(\sigma)(1-\beta^2\sigma\theta)^2}{(1-\beta^2\sigma^2)^2}\right]}{d(\beta\sigma)} \tag{A2}$$

The integral term of the expression (A2) is improper, however, and must be evaluated at the singular point $\theta = \sigma$. If the expression (A2) is now integrated by parts, account being taken of the singular point, there is obtained with $z = 0$

$$\lim_{\eta\to 0}\left\{\sqrt{1-\beta^2\theta^2}\int_{-\beta C}^{\beta(\theta-\eta)}\left[\frac{\beta f(\sigma)(1-\beta^2\sigma\theta)^2}{(1-\beta^2\sigma^2)^2(\beta\sigma-\beta\theta)^2}\,d(\beta\sigma)\right] + \sqrt{1-\beta^2\theta^2}\int_{\beta(\theta+\eta)}^{\beta C}\left[\frac{\beta f(\sigma)(1-\beta^2\sigma\theta)^2}{(1-\beta^2\sigma^2)^2(\beta\sigma-\beta\theta)^2}\,d(\beta\sigma)\right] - \frac{2f(\theta)\sqrt{1-\beta^2\theta^2}}{\eta}\right\} \tag{A3}$$

Equation (12) may now be rewritten for w/x with $z = 0$:

$$\frac{w}{x} = \lim_{\eta \to 0} \left\{ \int_{-\beta C}^{\beta(\theta-\eta)} \left[\frac{\beta f(\sigma)\sqrt{1-\beta^2\theta^2}\left(1-\beta^2\sigma\theta\right)^2}{\left(1-\beta^2\sigma^2\right)^2(\beta\sigma-\beta\theta)^2} d(\beta\sigma) - \frac{3\beta f(\sigma)\left(1-\beta^2\sigma\theta\right)\coth^{-1}\zeta}{\left(1-\beta^2\sigma^2\right)^{5/2}} d(\beta\sigma) \right. \right.$$

$$\left. + \frac{2\beta f(\sigma)\sqrt{1-\beta^2\theta^2}}{(1-\beta^2\sigma^2)^2} d(\beta\sigma) \right] + \int_{\beta(\theta+\eta)}^{\beta C} \left[\frac{\beta f(\sigma)\sqrt{1-\beta^2\theta^2}(1-\beta^2\sigma\theta)^2}{(1-\beta^2\sigma^2)^2(\beta\sigma-\beta\theta)^2} d(\beta\sigma) \right.$$

$$\left. \left. - \frac{3\beta f(\sigma)(1-\beta^2\sigma\theta)\coth^{-1}\zeta}{(1-\beta^2\sigma^2)^2} d(\beta\sigma) + \frac{2\beta f(\sigma)\sqrt{1-\beta^2\theta^2}}{(1-\beta^2\sigma^2)^2} d(\beta\sigma) \right] - \frac{2f(\theta)\sqrt{1-\beta^2\theta^2}}{\eta} \right\} \qquad \text{(A4)}$$

Following Leibnitz' rule for differentiation under the integral sign and collecting terms gives finally:

$$
\begin{aligned}
\frac{\partial(w/x)}{\partial\theta} = \lim_{\eta\to 0}\Bigg\{ & \int_{-\beta C}^{\beta(\theta-\eta)}\Bigg[\frac{3\beta^3\sigma f(\sigma)\coth^{-1}\zeta}{(1-\beta^2\sigma^2)^{5/2}}\,d(\beta\sigma) - \frac{\beta^2(3\beta\sigma + 2\beta\theta + \beta\theta\beta^2\sigma^2)f(\sigma)}{\sqrt{1-\beta^2\theta^2}(1-\beta^2\sigma^2)^2}\,d(\beta\sigma) \\
& + \frac{\beta\theta\beta^2 f(\sigma)}{\sqrt{1-\beta^2\theta^2}(\beta\sigma-\beta\theta)^2}\,d(\beta\sigma) - \frac{\beta^2 f(\sigma)}{\sqrt{1-\beta^2\theta^2}(1-\beta^2\sigma^2)(\beta\sigma-\beta\theta)}\,d(\beta\sigma) + \frac{2\beta^2 f(\sigma)\sqrt{1-\beta^2\theta^2}}{(\beta\sigma-\beta\theta)^3}\,d(\beta\sigma)\Bigg] \\
& + \int_{\beta(\theta+\eta)}^{\beta C}\Bigg[\frac{3\beta^3\sigma f(\sigma)\coth^{-1}\zeta}{(1-\beta^2\sigma^2)^{5/2}}\,d(\beta\sigma) - \frac{\beta^2(3\beta\sigma + 2\beta\theta + \beta\theta\beta^2\sigma^2)f(\sigma)}{\sqrt{1-\beta^2\theta^2}(1-\beta^2\sigma^2)^2}\,d(\beta\sigma) \\
& + \frac{\beta\theta\beta^2 f(\sigma)}{\sqrt{1-\beta^2\theta^2}(\beta\sigma-\beta\theta)^2}\,d(\beta\sigma) - \frac{\beta^2 f(\sigma)}{\sqrt{1-\beta^2\theta^2}(1-\beta^2\sigma^2)(\beta\sigma-\beta\theta)}\,d(\beta\sigma) + \frac{2\beta^2 f(\sigma)\sqrt{1-\beta^2\theta^2}}{(\beta\sigma-\beta\theta)^3}\,d(\beta\sigma)\Bigg] \\
& - \frac{2\beta\theta f(\theta)}{\eta\sqrt{1-\beta^2\theta^2}} - \frac{4\sqrt{1-\beta^2\theta^2}f'(\theta)}{\eta}\Bigg\} \qquad \text{(A5)}
\end{aligned}
$$

The second differentiation now gives

$$\frac{\partial^2(w/x)}{\partial\theta^2} = \lim_{\eta\rightarrow 0}\left\{6\sqrt{1-\beta^2\theta^2}\int_{-\beta C}^{\beta(\theta-\eta)}\frac{\beta^3 f(\sigma)}{(\beta\sigma-\beta\theta)^4}\,d(\beta\sigma) + 6\sqrt{1-\beta^2\theta^2}\int_{\beta(\theta+\eta)}^{\beta C}\frac{\beta^3 f(\sigma)}{(\beta\sigma-\beta\theta)^4}\,d(\beta\sigma)\right.$$

$$\left. - 4\sqrt{1-\beta^2\theta^2}\left[\frac{f''(\theta)}{\eta}+\frac{f(\theta)}{\eta^3}\right]\right\} \tag{A6}$$

The same process may be carried through for an incompressible, two-dimensional flow. The potential of a single doublet at a point $(y_1,0)$ in a two-dimensional field (y,z) would be (reference 8)

$$\phi = \frac{z}{(y_1-y)^2+z^2} \tag{A7}$$

from which w_1, the velocity normal to a flat plate extending along the y-axis from $-C$ to C, would be

$$w_1 = \int_{-C}^{C} f(y_1)\,dy_1\left[\frac{1}{(y_1-y)^2+z^2} - \frac{2z^2}{(y_1-y)^2+z^2}\right] \tag{A8}$$

Integrating by parts, then setting $z = 0$ as in equations (A1) to (A4) gives for $z = 0$

$$w_1 = \lim_{\eta\rightarrow 0}\left\{\int_{-C}^{(y-\eta)}\frac{f(y_1)\,dy_1}{(y_1-y)^2} + \int_{(y+\eta)}^{C}\frac{f(y_1)\,dy_1}{(y_1-y)^2} - \frac{2f(y)}{\eta}\right\} \tag{A9}$$

Differentiating twice with respect to y gives

$$\frac{\partial^2 w_1}{\partial y^2} = \lim_{\eta \to 0} \left\{ \int_{-C}^{(y-\eta)} \frac{6f(y_1)\,dy_1}{(y_1 - y)^4} + \int_{(y+\eta)}^{C} \frac{6f(y_1)\,dy_1}{(y_1 - y)^4} - \frac{4f(y)}{\eta^3} - \frac{4f''(y)}{\eta} \right\} \tag{A10}$$

This equation, except for the factor $\sqrt{1 - \beta^2\theta^2}$, is analogous to equation (A6). When the boundary conditions require the term $\frac{\partial^2(w/x)}{\partial\theta^2}$ to be zero, the factor may be omitted and solutions of equation (A10) are then seen to be solutions of equation (A6).

REFERENCES

1. Brown, Clinton E.: Theoretical Lift and Drag of Thin Triangular Wings at Supersonic Speeds. NACA TN No. 1183, 1946.

2. Ribner, Herbert S.: The Stability Derivatives of Low-Aspect-Ratio Triangular Wings at Subsonic and Supersonic Speeds. NACA TN No. 1423, 1947.

3. Puckett, Allen E.: Supersonic Wave Drag of Thin Airfoils. Jour. Aero. Sci., vol. 13, no. 9, Sept. 1946, pp. 475-484.

4. Prandtl, L.: Theorie des Flugzeugtragflügels im zusammendrückbaren Medium. Luftfahrtforschung, Bd. 13, Nr. 10, Oct. 12, 1936, pp. 313-319.

5. Theodorsen, Theodore: General Theory of Aerodynamic Instability and the Mechanism of Flutter. NACA Rep. No. 496, 1935.

6. de Haan, D. Bierens: Nouvelles Tables D'Intégrales Définies. G. E. Stechert & Co. (New York), 1939, pp. 474-476.

7. Stewart, H. J.: The Lift of a Delta Wing at Supersonic Speeds. Quarterly Appl. Math., vol. IV, no. 3, Oct. 1946, pp. 246-254.

8. Glauert, H.: The Elements of Aerofoil and Airscrew Theory. Cambridge Univ. Press, reprint of 1937, pp. 88-93.

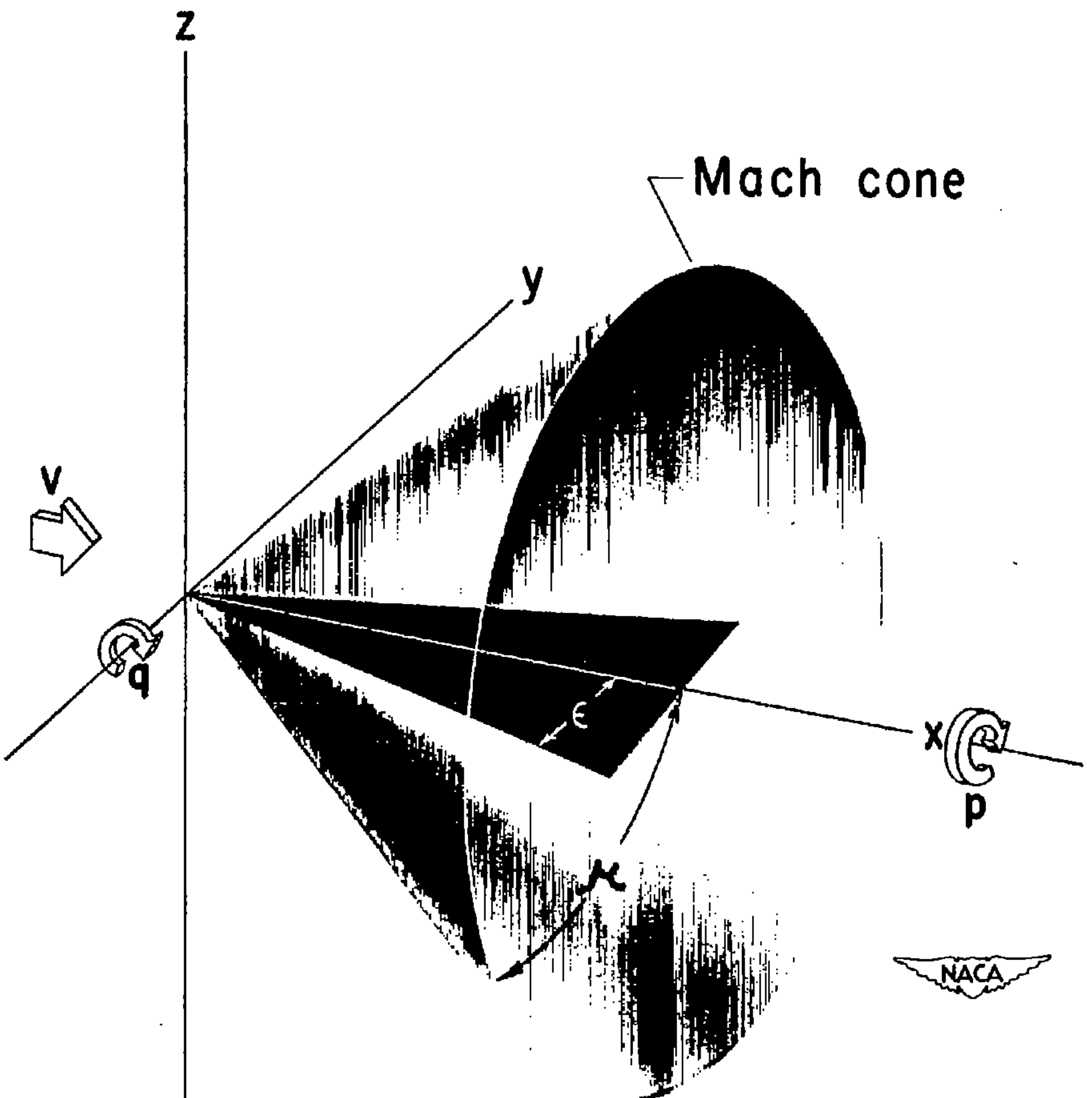

Figure 1.- Coordinate system.

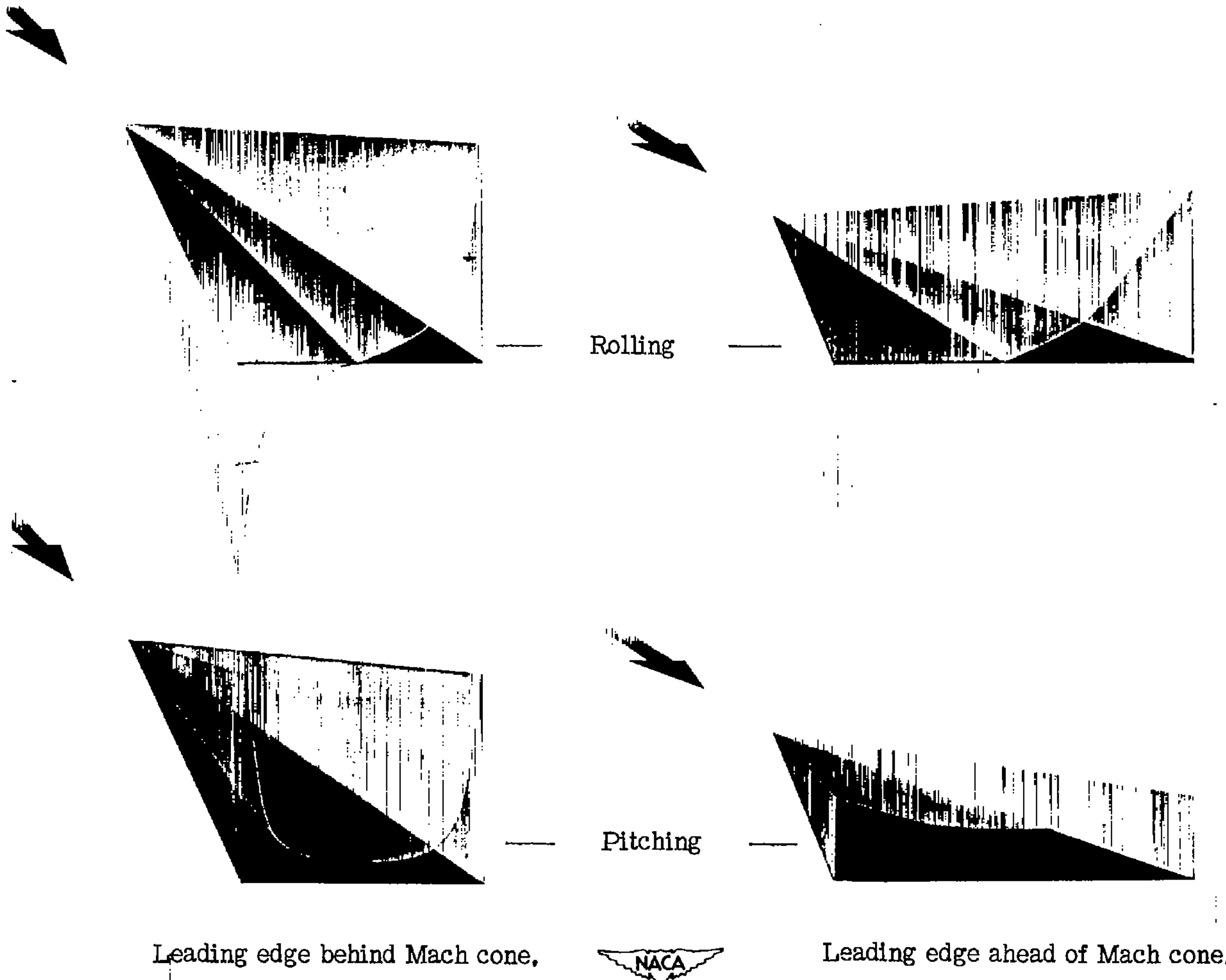

Figure 2.- Pressure distributions for rolling and pitching about apex.

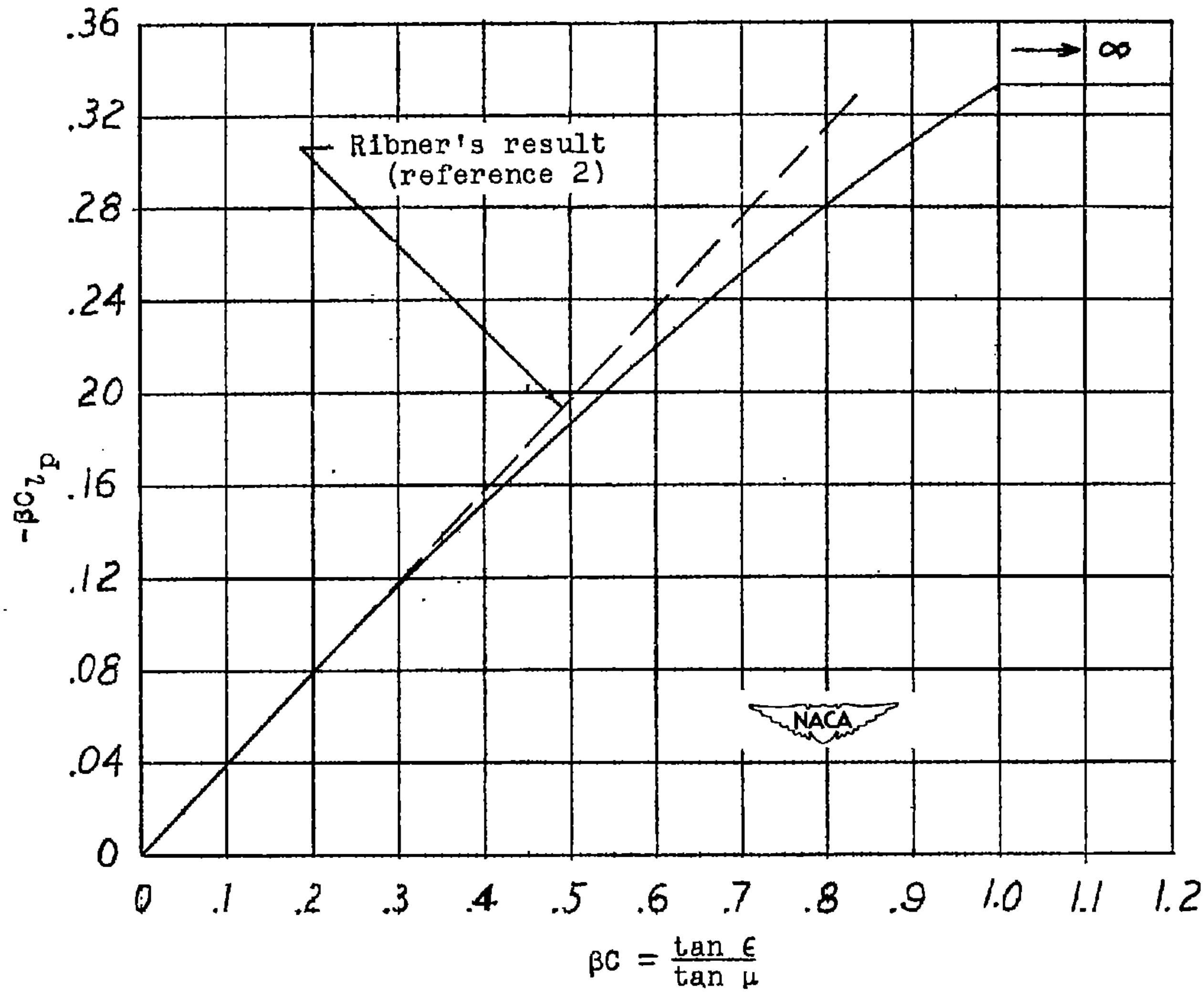

Figure 3.- Stability derivative C_{l_p} for triangular wings.

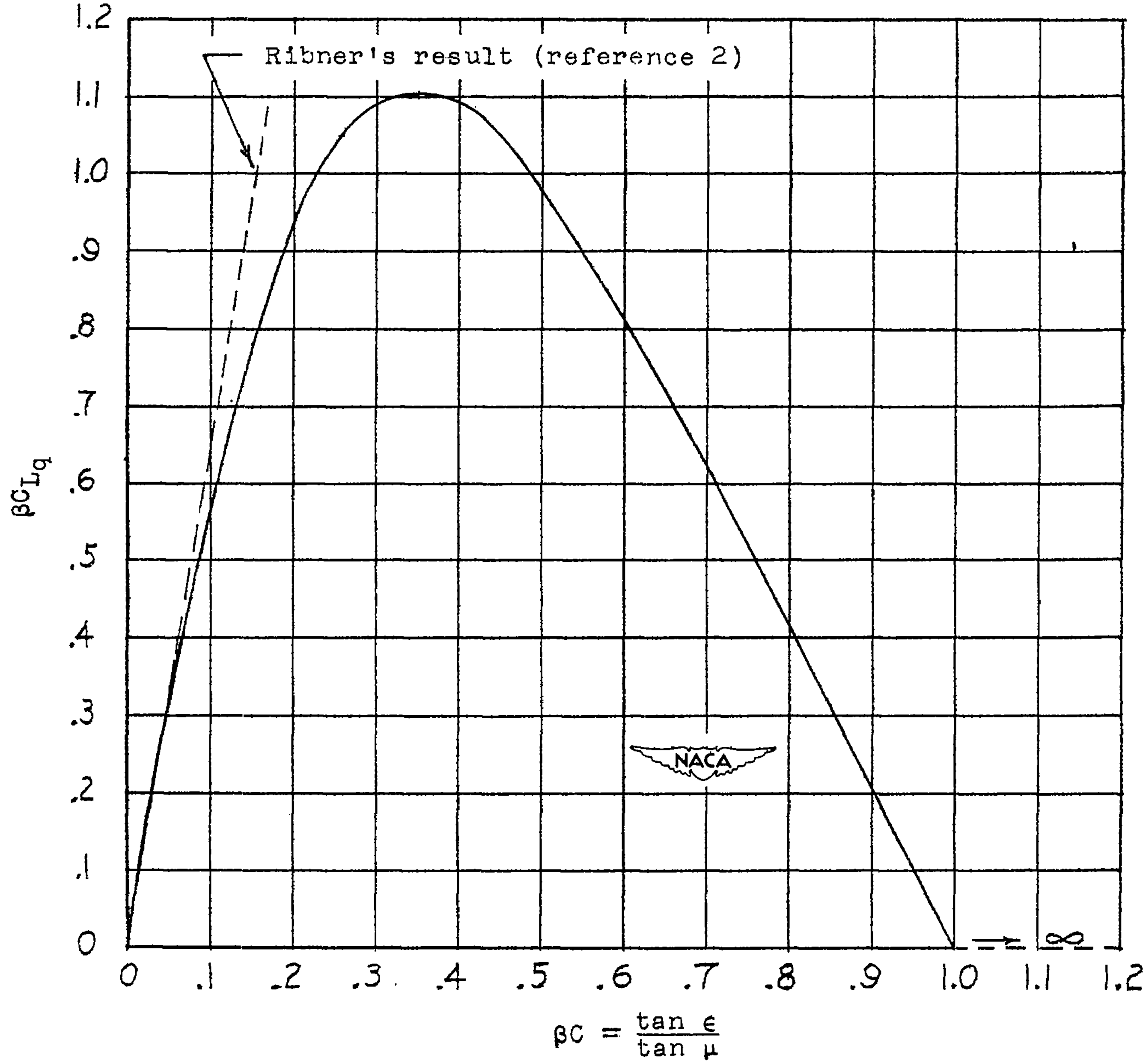

Figure 4.- Stability derivative C_{L_q} about the $\frac{2}{3}c$ point for triangular wings.

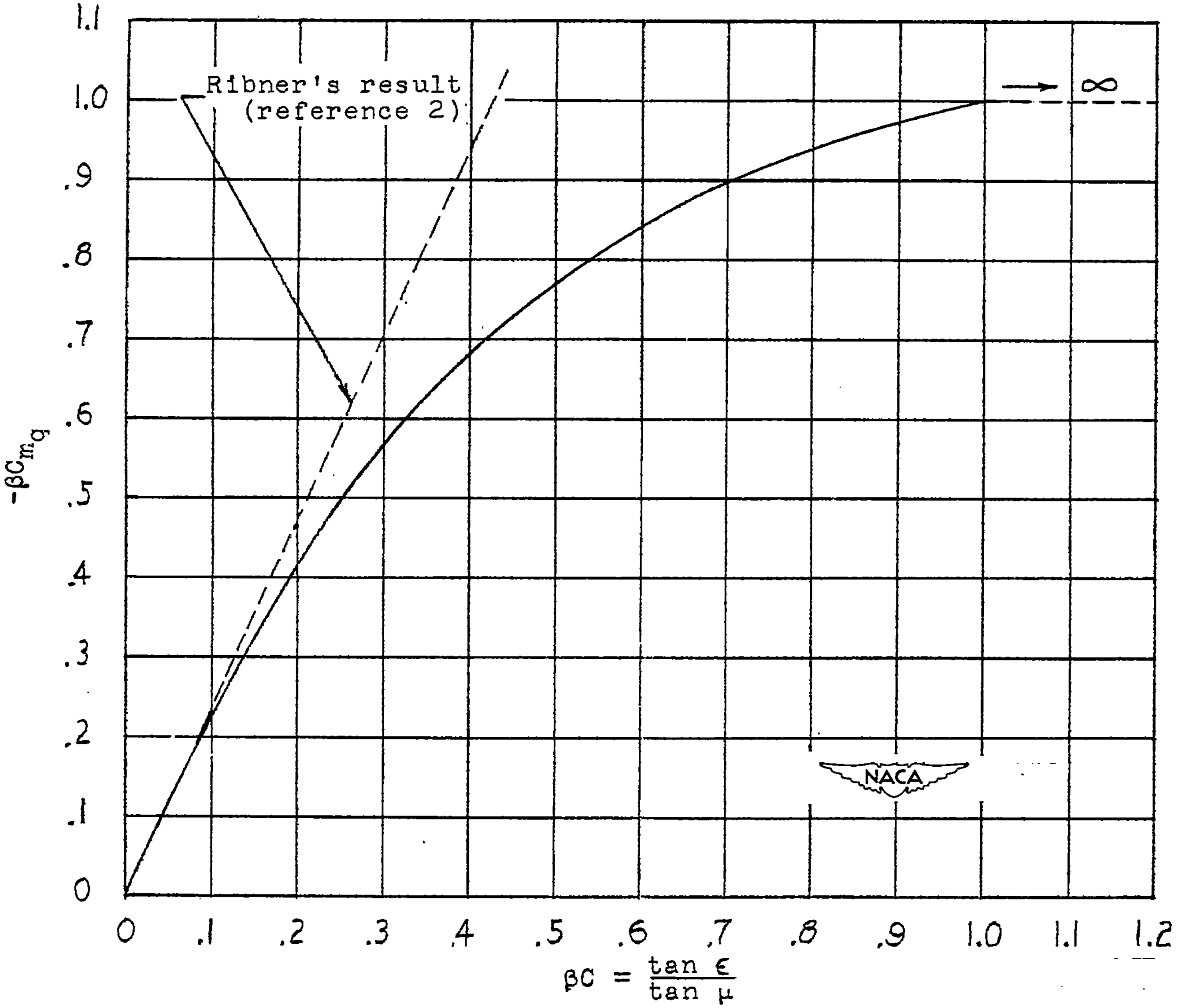

Figure 5.- Stability derivative C_{m_q} about the $\frac{2}{3}c$ point for triangular wings.